A Note on a General Two-Sector Model of Endogenous Growth with Physical and Human Capital

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October, 2000

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1. Introduction

As is well known, Investing resources into physical capital is the primary engine of economic growth. It is also recognized that investment into technological innovations plays an important role for the economy to grow. In other words, there is available an opportunity to invest economic resources into research and development activities in colleges and/or research organizations. Such an investment increases the quality of products, production efficiency or the number of products lines, and so it induces the economy to grow faster, other things being equal. This aspect is captured by models such as Aghion and Howitt(1992), Grossman and Helpman(1991) and Romer(1990). See Aghion and Howitt(1998) for the detailed exposition of Schumpeterian models.

It has been also emphasized that investment into human capital is an another important mechanics to increase the growth rate of the economy. The labor force usually consists of many varieties of services, which varies from manual labor through skilled and intelligent labor. As the larger fraction of population is educated in the colleges or professional schools, the fraction of intelligent and professional workers in the population becomes larger. That is, investment into education and job training programs makes the human capital to increase. The marginal product of labor increases as human capital accumulates even if physical capital stock and the labor population remain constant. This insight is captured in such models developed by Lucas(1988), Rebelo(1991) and others. See Barro and Sala-i-Martin(1995) for developments in the theoretical and empirical studies of endogenous growth.

This note develops a general two-sector endogenous growth model with physical and human capital and explores the conditions for an endogenous growth path to exist. Our analysis attempts to extend the results derived in the recent literature of endogenous growth models. The general two-sector model is presented in Section 2. The definition for the balanced growth path is given in Section 3 and then we will devote our efforts to deriving the conditions for the balanced growth path with nonzero growth rates to be sustained in Section 4. Section 5 concludes the paper.

2. The Model

We suppose that there is a single physical goods which can be consumed as the consumption goods and can be converted into investment goods. This physical goods are produced by combining physical capital, unskilled labor services, and skilled human capital. The production technology of the physical (consumption) goods sector is characterized by the following production function:

(1) $Y = f^{p}(K_1, L_1, h) f^{e}(K_a, H_a)$

where Y is the total output of consumable goods, f^p is the private production, f^e captures the externality effect arising from the increase in physical and/or human capital. K_1 is the amount of physical capital used in the physical goods sector, and L_1 is the amount of labor employment in this sector. H and h have two interpretations. One interpretation is to assume that H is the total amount of human capital and h is the human capital per head. Under this interpretation, the output level depends on both the number of labor employment and the average level of human capital per head. The popular endogenous growth model specializes in the case in which the production function is to be $f^{p}(K_1, L_1 h)$. Our formulation extends this special case into more general cases. The other interpretation is to take H to be the knowledge produced by R&D activity. In this case, our specification for the production function captures the property that the production level depends on the industry-wide average index of total knowledge accumulated in the economy.

We assume that human capital increases as the amount of knowledge and/or know-how ,D, embodied in the labor population expands. The production activity in the knowledge production sector increases the amount of knowledge or know-how embodied in the labor force. The output D in the knowledge sector is produced by the following technology:

(2) $D = g^{p}(K_2, L_2, h) g^{e}(K_a, H_a)$

where g^p is the private production function, g^e is the function reflecting the externality effect, and the subscript 2 stands for the knowledge production sector.

The total number of labor force measured by the number of heads, denoted by *L*, is assumed to grow at the constant rate *n*. The human capital per head is defined by $h = H/L$.

By K denote the total amount of physical capital. The full utilization of physical capital is expressed by

$$K = K_1 + K_2.$$

Similarly the full employment condition for labor is

$$L = L_1 + L_2.$$

We assume that the production functions f^p, g^p are homogenous of the degree one with respect to capital stock K and labor employment L. This implies that the increasing returns to scale prevails while the constant returns to scale holds given the value of average human capital h.

It is convenient to summarize here the assumptions imposed upon the production functions and utility function.

The Assumption 1:

(a) The production functions f^p and g^p are increasing and concave with respect to each argument.

(b) f^p and g^p are continuous and C^2.

The homogeneity assumption has been imposed in most models in the literature on endogenous growth theories. For instance, it is assumed that f^p and g^p are homogenous of the degree one in physical and labor employment, K_1 and L_1, or linearly homogeneous in physical capital and human capital. It could be imposed that f^p is homogenous of the degree one for all arguments *(k, l, h)*. It should be remarked, however, that the bounded property assumption given by Assumption 3 in the appendix must be satisfied.

The Assumption 2:

(a) The utility function *U(c)* is increasing and concave.

(b) U(c) is continuous and C^2.

These are the crucial assumptions, which are assumed in the all models of endogenous growth.

We need to introduce the per head variables as follows:

$$y = Y/L$$
$$k = K/L$$
$$d = D/L$$

Denote by u and v , respectively, the fraction of physical capital and the fraction of labor force employed in the physical goods sector. Then the following relationships must hold:

$$k_1 = uk, k_2 = (1-u)k,$$

$$L_1 = vL, L_2 = (1-v)L,$$

$$k_1 = uk, k_2 = (1-u)k,$$

$$0 \le u \le 1, 0 \le v \le 1$$

We shall begin our analysis from the simplest case so that the externality is ignored. The suprescript p on the private production functions will be dropped below without confusion. Using these variables the output of physical goods per head and the output of knowledge goods per head are given by

(3) $y = f^p(uk, v, h) f^e$

(4) $d = g^p((1-u)k, 1-v, h) g^e,$

where $f^e = 1,\ g^e = 1$.

Denote by δ_1 the rate of depreciation for physical capital and by δ_2 , its rate for human capital. We denote by c the consumption per head. From the relationships above we have the accumulation equation for physical goods:

(5) $$\frac{dk(t)}{dt} = f^p(uk, v, h) f^e - c(t) - (n + \delta_1) k(t),$$

where t stands for calendar time. Similarly the accumulation of human capital is governed by

(6) $$\frac{dh(t)}{dt} = g^p((1-u)k, 1-v, h) g^e - (n + \delta_2) h(t).$$

The optimization problem faced by the household is to choose the time stream of the allocations for physical capital and labor force between the two sectors, *{u(t),v(t)}*, and the time profile of future consumption *{c(t)}* that maximizes the discounted present value of the future utilities

(7) $J(k_0, h_0, u, v, c) = \int_0^{\infty} e^{-(\rho - n)t} U(c(t)) dt$,

subject to the accumulation equations (5) and (6), given the initial values of physical and human capital. Here ρ is the marginal rate of time preference, k_0 and h_0 are the initial values for the physical capital and human capital.

The state variables for the present optimization problem are the per capita physical capital and the human capital. We assume that the state vector lies in the vector space X of all piecewise C^1 functions with values in R^2 defined on a given interval within [0,∞). Let X denote the subset consisting of those *(k, h)*$\in X$ which satisfy the initial conditions $k(0) = k_0, h(0) = h_0$. The control vector consists of *u(t), v(t),* and *c(t),* which are real-valued piecewise C^1 functions from [0,∞) to non-negative real value and must satisfy the restrictions, $0 \le u \le 1, 0 \le v \le 1.$ The instantaneous value of consumption is bounded above because it must bet true that $f^p(uk, v, h) \ge c(t)$. Therefore the control variables is the convex bounded subset of R^3, $\Omega = \{u, v, c{:}\, 0 \le u \le 1, 0 \le v \le 1, \text{ c is bounded}\}$. Let M be a set of piecewise continuous functions $\{u(t), v(t), c(t)\}$ with values in Ω, defined on [0,∞). Denote by F the class of all functions (k_0, h_0, u, v, c) such that (*u,v,c*) is a Lebesgue-integrable functions on R with values in Ω and the solution of eqs.(5) and (6) satisfies the initial conditions. The optimization problem is to find in the class F an element (k_0, h_0, u, v, c) such that the discounted present value of future stream of utilities $J(k_0, h_0, u, v, c)$ is maximized.

3. The Derivation of the Optimal Solution

The maximization problem stated above can be solved by the Maximum Principle of Pontryagin. First, consider the finite-time version of the present model: Maximize

(7') $J(k_0, h_0, u, v, c, T) = \int_0^{T} e^{-(\rho - n)t} U(c(t)) dt$

subject to (5) and (6), where the terminal state is free. To solve this problem, define the Hamiltonian function H :

$$H(t,k,h,u,v,c) = e^{-(\rho-n)t}[U(c(t)) + q_1(t)\{f^p(u(t)k(t),v(t),h(t)) - c(t) - (n+\delta_1)k(t)\} + q_2(t)\{g^p((1-u(t))k(t),1-v(t),h(t)) - (n+\delta_2)h(t)\}] \quad (8)$$

where q_1 and q_2 are costate variables corresponding to eqs.(5) and (6). The necessary conditions that (u^*,v^*,c^*) is the optimal solution are given by this: There exists a nonzero two-dimensional vector function (q_1,q_2) such that

$$\frac{dq_1(t)}{dt} = -\frac{\partial H(t,k^*,h^*,u^*,v^*,c^*)}{\partial k} , \quad (9)$$

$$\frac{dq_2(t)}{dt} = -\frac{\partial H(t,k^*,h^*,u^*,v^*,c^*)}{\partial h}, \quad (10)$$

$$\max_{(u,v,c)\in\Omega} H(t,k^*,h^*,u,v,c) = H(t,k^*,h^*,u^*,v^*,c^*). \quad (11)$$

The transversality condition is given by

$$\begin{aligned} &e^{-(\rho-n)T}q_1(T)k(T) = 0 \\ &e^{-(\rho-n)T}q_2(T)h(T) = 0 \end{aligned} . \quad (12)$$

As shown in the Appendix, there exists an potimal solution for this optimization problem. There exists an optimal solution for any T>0, so that there is a limiting sequence for (u,v,c) as T goes to infinity. We take this limit point to be the optimal solution for the original infinite-time optimization problem. The transversality condition is then given by

$$\lim_{t\to\infty} e^{-(\rho-n)t}q_1(t)k(t) = 0$$

$$\lim_{t\to\infty} e^{-(\rho-n)t}q_2(t)h(t) = 0$$

Now we characterize the dynamic property of the solution.

The condition (11) is equivalent to

$$\frac{\partial H(t,k^*,h^*,u^*,v^*,c^*)}{\partial u} = 0,$$

$$\frac{\partial H(t,k^*,h^*,u^*,v^*,c^*)}{\partial v} = 0,$$

$$\frac{\partial H(t,k^*,h^*,u^*,v^*,c^*)}{\partial c} = 0.$$

The explicit expression of these equations is given by

$$q_1(t)\frac{\partial f^p(u(t)k(t),v(t),h(t))}{\partial u(t)} = -q_2(t)\frac{\partial g^p((1-u(t))k(t),1-v(t),h(t))}{\partial u(t)}, \quad (12a)$$

$$q_1(t)\frac{\partial f^p(u(t)k(t),v(t),h(t))}{\partial v(t)} = -q_2(t)\frac{\partial g^p((1-u(t))k(t),1-v(t),h(t))}{\partial v(t)}, \quad (12b)$$

(12c) $U'(c(t)) = q_1(t).$

These relationships determine the values of u(t), v(t), c(t) as the functions of the state vector and costate vector. Eqs.(9) and (10) are expressed in the following form:

(13a) $$\frac{dq_1(t)}{dt} = -q_1(t)\frac{\partial f^p(u(t)k(t),v(t),h(t))}{\partial k(t)} - q_2(t)\frac{\partial g^p((1-u(t))k(t),1-v(t),h(t))}{\partial k(t)} + (\rho+\delta_1)q_1(t),$$

(13b) $$\frac{dq_2(t)}{dt} = -q_1(t)\frac{\partial f^p(u(t)k(t),v(t),h(t))}{\partial h(t)} - q_2(t)\frac{\partial g^p((1-u(t))k(t),1-v(t),h(t))}{\partial h(t)} + (\rho+\delta_2)q_2(t).$$

The optimal growth path is described by the four differential equations, (5), (6), (13a) and (13b) combined with eqs.(12a,b,c).

From (12a) and (12b), we have

(12a') $q_1(t)\, f_k^p(u(t)k(t),v(t),h(t)) = q_2(t)g_k^p((1-u(t))k(t),1-v(t),h(t)),$

(12b') $q_1(t)\, f_l^p(u(t)k(t),v(t),h(t)) = -q_2(t)g_l^p((1-u(t))k(t),1-v(t),h(t)),$

where subscripts k ,l and h stand for the partial derivative of the relevant functions with respect to the first, second, and third argument, respectively. It is straightforward to see that

(14a) $$\frac{f_k^p(u(t)k(t),v(t),h(t)}{f_l^p(u(t)k(t),v(t),h(t))} = \frac{g_k^p((1-u(t))k(t),1-v(t),h(t))}{g_l^p((1-u(t))k(t),1-v(t),h(t))}$$

which means that the marginal rate of substitution between the physical capital and labor is equalized between the two sectors. Solving this for v yields

(14b) $v(t) = v(u(t),k(t),h(t)).$

Substituting (12a,b) into (13a) and (13b), we obtain

(15a) $$\frac{dq_1(t)}{dt} = -q_1(t)\, f_k^p(u(t)k(t),v(t),h(t)) + (\rho+\delta_1)q_1(t),$$

(15b) $$\frac{dq_2(t)}{dt} = -q_2(t)\frac{g_k^p((1-u(t))k(t),1-v(t),h(t))}{f_k^p((1-u(t))k(t),1-v(t),h(t))}\, f_h^p(u(t)k(t),v(t),h(t)) - \mathrm{q}_2(\mathrm{t})\, g_h^p((1-u(t))k(t),1-v(t),h(t)) + (\rho+\delta_2)q_2(t).$$

The time trajectory of equilibrium growth path is completely determined by the system of differential equations (5), (6), (15a) and (15b) with (14a) and (14b).

The balanced growth path can be defined as follows:

Definition(A): Among the time trajectories of the state and costate vectors solving the dynamical system described above, an optimal path is called the balanced growth path if along its path there exists such real numbers $\{\gamma_k, \gamma_h, \gamma_1, \gamma_2, \gamma_c\}$ that

for all $t \geq 0$, $\dot{k}/k = \gamma_k \geq 0, \dot{h}/h = \gamma_h \geq 0, \dot{q}_1/q_1 = \gamma_1 \leq 0, \dot{q}_2/q_2 = \gamma_2 \leq 0, \dot{c}/c = \gamma_c \geq 0$

, and furthermore $u(t) = \text{constant}, v(t) = \text{constant}$, where the dot (.) over the variables refers to the time derivatives.

According to this definition, along the balanced growth path the ratio of consumption over physical capital may not remain constant, which implies that the output/physical-capital might vary also over time. However, the ratio of investment over physical capital remains constant. In the standard definition of the balanced growth path the constancy of the ratio of consumption over physical capital is assumed. The implication of this difference is explored later. The steady state of dynamical system under consideration obtains when the conditions $\gamma_k = 0, \gamma_h = 0, \gamma_1 = 0, \gamma_2 = 0$ hold. This corresponds to the balanced growth path with zero growth-rate. The balanced growth path with non-zero growth rates corresponds to the so-called endogenous growth path.

Let denote by μ_1 and μ_2 the parameters which characterize the production function for physical goods and knowledge goods, respectively. We denote this property by

$$f^p(uk, v, h; \mu_1), g^p((1-u)k, (1-v), h; \mu_2).$$

For instance, simple examples are

(CB) $$f^p(uk, v, h; \mu_1) = (uk)^{\alpha_1} v^{\beta_1} h^{\mu_1},$$
$$g^p((1-u)k, (1-v), h; \mu_2) = \{(1-u)k)^{\alpha_2} (1-v)^{\beta_2} h^{\mu_2}.$$

It is well known that the vector field generated by the dynamical system changes qualitatively as the value of system parameters μ_1 and / or μ_2 crosses the bifurcation point on the parameter space and this bifurcation point is closely related to the existence condition of the endogenous growth path.

4. The Condition for the Balanced Growth to Exist

From (12c) we have

(16) $-\sigma\gamma_c = \gamma_1$,

where σ is the elasticity of marginal utility

$$\sigma = -\frac{U''(c)}{U'(c)}c.$$

For the growth rate of consumption and the shadow price of capital to be sustained, the elasticity of marginal utility must be constant. The common practice in the literature assumes that the utility function is given by the specific functional form:

$$U(c) = \frac{c^{1-\theta} - 1}{1-\theta}, \sigma = 1/\theta$$

See Caballe and Santos(1993) as for the similar argument in simpler models. To transform the original dynamical system into the convenient reduced form, define new variable x by

$$x = \frac{c}{k},$$

which implies that

$$\frac{\dot{x}}{x} = \frac{\dot{c}}{c} - \frac{\dot{k}}{k}.$$

Substituting (5), (15a) and (16) into this equation, we have

(17a) $$\dot{x} = [\frac{f_k - (\rho + \delta_1)}{\sigma} - \{\frac{f}{k} - x - (n+\delta_1)\}]x.$$

To have the dynamic equation for u, take the time derivative of (12a').

(17b) $$\{\frac{f_{kk}}{f_k}k + \frac{g_{kk}}{g_k}k + \frac{f_{kl}}{f_k}v' + \frac{g_{kl}}{g_k}v'\}\dot{u} = f_k - \frac{g_k}{f_k}f_h - g_h + \delta_2 - \delta_1$$
$$+ \{\frac{g_{kk}}{g_k}(1-u) - \frac{f_{kk}}{f_k}u\}\{f - xk - (n+\delta_1)k\}$$
$$+ \{\frac{g_{kh}}{g_k} - \frac{f_{kh}}{f_k}\}\{g - (n+\delta_2)h\}.$$

The accumulation equations for physical and human capital can be rewritten by

(17c) $$\dot{k} = f - xk - (n+\delta_1)k,$$

(17d) $$\dot{h} = g - (n+\delta_2)h.$$

The dynamical system composed of (17a) through (17d) completely describe the time evolution of the competitive growth path. The fixed point of this dynamical system if it exists, must satisfy

(18a) $$f = xk + (n+\delta_1)k,$$

(18b) $$g = (n+\delta_2)h,$$

(18c) $$f_k = \rho + \delta_1,$$

(18d) $$\frac{g_k}{f_k}f_h + g_h = \rho + \delta_2.$$

This system of equations characterizes the property of the zero-growth steady state. The economic meaning of these conditions is clearly understood as in the standard capital theory. The left hand side of the last equation can be interpreted as the marginal product of human capital in the knowledge industry. The difference between the present model and the standard model in capital theory lies in the fact that in our model the bifurcation occurs as the value of system parameters μ_1 and μ_2 varies and this bifurcation corresponds to the occurrence of the endogenous growth.

We need to derive the condition for this bifurcation to occur. We can prove the proposition below.

Proposition 1:

For the balanced growth path with nonzero-growth rate to be sustained, there must exist positive constants (γ_k, γ_h) and constants (γ_x, γ_{12}) that satisfy the following relationships:

(19a) $\frac{f}{k} - x - (n+\delta_1) = \gamma_k$,

(19b) $\frac{g}{h} - (n+\delta_2) = \gamma_h$,

(19c) $f_k = \rho + \delta_1 + \sigma(\gamma_x + \gamma_k)$,

(19d) $\frac{g_k}{f_k} f_h + g_h = \rho + \delta_2 + \frac{\gamma_x + \gamma_k}{\sigma} + \gamma_{12}$,

(19e) $\{\frac{g_{kk}}{g_k}(1-u) - \frac{f_{kk}}{f_k} u\} k\gamma_k + (\frac{g_{kh}}{g_k} - \frac{f_{kh}}{f_k}) h\gamma_h - \gamma_{12} = 0$,

where γ_x is the growth rate of x and $\gamma_{12} = \gamma_1 - \gamma_2$.

Proof: From (17c) and (17d), it is obvious that (19a) and (19b) must hold. From (17a) and (17b) the following relationships must hold:

(20a) $\frac{f_k - (\rho + \delta_1)}{\sigma} - \{\frac{f}{k} - x - (n+\delta_1)\} = \gamma_k$,

(20b) $f_k - \frac{g_k}{f_k} f_h - g_h + \delta_2 - \delta_1 + \{\frac{g_{kk}}{g_k}(1-u) - \frac{f_{kk}}{f_k} u\} k\gamma_k + \{\frac{g_{kh}}{g_k} - \frac{f_{kh}}{f_k}\} h\gamma_h = 0$.

Combining (19a) and (20a) yields (19c). From (15a) and (15b) we have

(20c) $\gamma_{12} = \gamma_1 - \gamma_2 = -f_k + \frac{g_k}{f_k} f_h + g_h + \delta_1 - \delta_2$.

Substituting (19c) into (20c) leads to (19d). Plugging (20c) into (20b) we have (19e). This completes the proof.

(19a) shows that the growth rate of physical capital must be constant over time and (19b) means that the stock of knowledge must accumulate with constant growth rate. (19c) implies that the marginal product of physical capital in the physical goods sector remains constant. The left hand side of (19d) is interpreted as the marginal product of human capital in the knowledge sector. (19d) shows that the marginal product of human capital must remain constant over time. (19e) is the condition for temporal allocation of physical capital and labor between the two sectors to remain constant over time.

We can prove the next proposition:

Proposition 2:

The transversality condition for the balanced growth path is satisfied if and only if the following inequalities hold:

(21a) $\frac{f-c}{k} - f_k < 0,$

(21b) $\frac{g}{h} - (\frac{g_k}{f_k} f_h + g_h) < 0.$

Proof: The growth rates of physical and human capital, costate variables must be constant along the balanced growth path, which are given by eqs.(15a,b) and (17c,d). We have

$$\gamma_k = \frac{f-c}{k} - (n+\delta_1),$$

$$\gamma_h = \frac{g}{h} - (n+\delta_2),$$

$$\gamma_1 = \rho + \delta_1 - f_k,$$

$$\gamma_2 = \rho + \delta_2 - (\frac{g_k}{f_k} f_h + g_h).$$

The transversality condition is equivalent to the inequalities:

$$-(\rho - n) + \gamma_1 + \gamma_k < 0,$$

$$-(\rho - n) + \gamma_2 + \gamma_h < 0.$$

Substituting the expression for growth rates yields (21a) and (21b).

To draw the economic implications of these propositions, consider the example where the production functions are specified by the Cob-Douglas functions:

(22) $f^{p}(uK, vL, h) = A(uK)^{\alpha_1}(vL)^{\beta_1}h^{\mu_1}, \alpha_1 \geq 0, \beta_1 > 0, \mu_1 > 0,$

(23) $g^{p}((1-u)K, (1-v)L, h) = B\{(1-u)K\}^{\alpha_2}\{(1-v)L\}^{\beta_2}h^{\mu_2}, \alpha_2 \geq 0, \beta_2 > 0, \mu_2 > 0.$

The models used by Mulligan and Sala-i-Martin(1993) and Bond, Wang and Yip(1996) are the special case of this example. That is, it is assumed that $\mu_1 = \beta_1, \mu_2 = \beta_2$.

Under this example, eqs.(19a) through (19e) are simplified into

(24a) $\gamma_k = Au(uk)^{\alpha_1 - 1}v^{\beta_1}L^{\alpha_1+\beta_1-1}h^{\mu_1} - x - (n + \delta_1),$

(24b) $\gamma_h = B\{(1-u)k\}^{\alpha_2}(1-v)^{\beta_2}L^{\alpha_2+\beta_2-1}h^{\mu_2-1} - (n + \delta_2),$

(24c) $\alpha_1 Au(uk)^{\alpha_1-1}v^{\beta_1}L^{\alpha_1+\beta_1-1}h^{\mu_1} = \rho + \delta_1 + \sigma(\gamma_x + \gamma_k),$

(24d) $\{\mu_2 + \alpha_2 \dfrac{\mu_1 u}{\alpha_1(1-u)}\}B\{(1-u)k\}^{\alpha_2}(1-v)^{\beta_2}L^{\alpha_2+\beta_2-1}h^{\mu_2-1}$

$$= \rho + \delta_2(\gamma_x + \gamma_k)/\sigma + \gamma_{12},$$

(24e) $(\alpha_2 - \alpha_1)\gamma_k + (\mu_2 - \mu_1)\gamma_h - \gamma_{12} = 0.$

For the balanced growth path with nonzero-growth rate to be sustained, there must exist positive constants (γ_k, γ_h) and constants (γ_x, γ_{12}) that satisfy eqs.(24a) through (24e).

Suppose that the prodction functions are linearly homogeneous in physical capital and labor. This assumption is equivalent to the restriction:

$$\alpha_1 + \beta_1 = 1, \text{and } \alpha_2 + \beta_2 = 1.$$

Under this restriction, the necessary conditions for the balanced growth path with non-zero growth rates to exist can be derived by taking the time derivatives of (24c) and (24d). We will have

(25a) $(\alpha_1 - 1)\gamma_k + \mu_1\gamma_h = 0,$

(25b) $\alpha_2\gamma_k + (\mu_2 - 1)\gamma_h = 0.$

We can observe that when (25a) holds, the marginal product of physical capital in the physical goods sector remains constant, and so the first term on the right hand side of (24a) must be constant. This fact implies that the second term on the right hand side of (24a) must be constant, which means that the ratio of consumption over physical capital is constant along the balanced growth path. That is, (26a) $\gamma_c = \gamma_k$.

It is clearly understood that if the production function is Cob-Douglas type, the ratio of consumption over physical capital remains constant along the balanced growth path. When (25b) holds, the marginal product of human capital in the knowledge sector remains constant so that the first term on the right hand side of (24b) must be constant. It is clear that for the endogenous growth to be sustained, the both of the marginal product of physical capital in the physical goods sector and the marginal product of human capital in the knowledge sector must be constant. Substituting (25a) and (25b) into (24e) yields

(26b) $\gamma_k - \gamma_h + \gamma_{12} = 0.$

This relationship shows that the physical capital and human capital might not grow with the same growth rate unless $\gamma_{12} = 0.$ It implies that the ratio of consumption over human capital may not remain constant along the endogenous growth path. The condition $\gamma_{12} = 0$ is not necessarily required as a condition for the system of equations (24a) through (24e) to have non-negative solutions.

For the system of equations (25a) and (25b) to have a non-zero solution, it is necessary that

(27) $(\alpha_1 - 1)(\mu_2 - 1) = \alpha_2 \mu_1.$

Without the condition (27), there exists no endogenous growth path but there exists a balanced growth path with zero growth rate, in which per head physical capital and human capital remain constant over time. This condition is an extention of those given by Mulligan and Sala-i-Martin(1993) and Barro and Sala-i-Martin(1995,chapter 5).

As clearly shown above, the condition (27) does not implies that the growth rates of physical capital and human capital per head are the same along the endogenous growth path. When the growth rates of physical and human capital are idential, $\gamma_k = \gamma_h$. Substituting this relationship into (25a,b), we have

(28) $\alpha_1 + \mu_1 = 1$ and $\alpha_2 + \mu_2 = 1.$

We conclude here that for the endogenous growth to occur, it is necessary to have the condition (28). When (28) holds, the production functions must be specified by

(29a) $f^p(uK, vL, h) = A(uK)^{\alpha_1}(vhL)^{1-\alpha_1}$, (29b) $g^{p}((1-u)K,(1-v)L,h)=B\{(1-u)K\}^{\alpha_2}\{(1-v)hL\}^{1-\alpha_2}$.

We can assert that for the endogenous growth path to be sustained with the constant ratio of consumption over human capital, the production functions must have the specific forms given above. The model used by Rebelo(1991), Mulligan and Sala-i-Martin(1993) and Bond, Wang, and Yip(1996) has exactly the same form as given above for the production functions.

Next suppose that the production functions are not linearly homogenous in physical capital and labor, which implies that $\alpha_1+\beta_1\neq 1$, and $\alpha_2+\beta_2\neq 1$. In this case, the eqs.(25a,b) turns out to be

(25'a) $(\alpha_1-1)\gamma_k+\mu_1\gamma_h+(\alpha_1+\beta_1-1)n=0$,

(25'b) $\alpha_2\gamma_k+(\mu_2-1)\gamma_h+(\alpha_2+\beta_2-1)n=0$.

This is the generalization of the result given by (25a,b). Suppose that eq.(27) does not hold. The growth rates of per head physical and human capital must be the same and so $\gamma_h=\gamma_k$.

Substituting this into the above equations leads to

$$(\alpha_1+\mu_1-1)\gamma_k+(\alpha_1+\beta_1-1)n=0,$$

$$(\alpha_2+\mu_2-1)\gamma_k+(\alpha_2+\beta_2-1)n=0.$$

For this equation to have any solution, it must hold that

(30) $(\alpha_1+\mu_1-1)(\alpha_2+\beta_2-1)=(\alpha_1+\beta_1-1)(\alpha_2+\mu_2-1)$.

When the parameters for the production technologies satisfy (30), a balanced growth may be sustained. It is also clear that for the growth rate to be positive, the following inequalities must hold:

$$(\alpha_1+\mu_1-1)(\alpha_1+\beta_1-1)<0, (\alpha_2+\beta_2-1)(\alpha_2+\mu_2-1)<0.$$

It implies that the increasing returns to scale with respect to physical capital and the average human capital embodied in labor employed in the production must prevail as far as the production technology shows the decreasing returns to scale with respect to physical capital and the amount of labor.

From Eq.(24e), we have

$$(\alpha_2+\mu_2-\alpha_1-\mu_1)\gamma_k=\gamma_{12}.$$

The requirement that the shadow price of physical and human capital grows at the same rate, Using (24e), yields $\alpha_2 + \mu_2 = \alpha_1 + \mu_1$. Eq.(30) then leads to $\alpha_2 + \beta_2 = \alpha_1 + \beta_1$. These imply that the production technologies must be governed by

(31) $f^p = AK_1^{\alpha_1}(hL_1)^{\beta_1} h^{\mu}, g^p = BK_2^{\alpha_2}(hL_2)^{\alpha_1+\beta_1-\alpha_2} h^{\mu}$.

Consider the special case that the production function for consumption goods is linearly homogenous in physical capital, labor, and the average human capital. In this case, $\alpha_1 + \beta_1 + \mu = 1$. It is obvious, then, that the production function for education must be linearly homogenous in physical capital, the total human capital and the average human capital. That is, the production technology must be expressed in the specific form:

$$f^p = AK_1^{\alpha_1}(hL_1)^{1-\alpha_1-\mu} h^{\mu}, g^p = BK_2^{\alpha_2}(hL_2)^{1-\alpha_2-\mu} h^{\mu}, 0 < \alpha_1, \alpha_2 < 1, 0 < \mu < 1.$$

This specific functional form will be satsified only when an increase in the average human capital itself, just like an external effect, contributes to an advancement of the productivity efficiency.

5. Concluding Remarks

We have shown that for the endogenous growth path to be sustained, the production functions must have the specific forms, for instance, given by (29a,b) and (31) as much as the Cob-Douglas functions are assumed. The specific production functions obtain just when the system parameter μ_i crosses the bifurcation point, for exmple, $\mu_i{*} = \beta_i, i = 1,2$ on the parameter space. Otherwise, any endogenous growth path cannot be sustained.

The condition for endogenous growth is given, in general, by eq.(31). The degree of freedom for the system parameters increases. Policy analysis such as the effects of taxes on the growth rate can be extended in the present version of models. In other words, it is not necessary to presuppose the constant returns to scale with respect to physical and human capital when we conducts policy analysis on the endogenously growing economy.

The Appendix

We must rewrite the problem into the Mayer problem. Let us Introduce the new state vector $x = (x_1, x_2, x_3)$, where $x_1 = k, x_2 = h$ and new control vector $u = (u_1, u_2, u_3)$, where $u_1 = u, u_2 = v, u_3 = c$. Eqs.(5) and (6) can be rewritten by

(5') $\frac{dx_1(t)}{dt} = f_1(t, x(t), u(t)) \equiv f^p(uk, v, h) - c(t) - (n+\delta_1)k(t),$

(6') $\frac{dx_2(t)}{dt} = f_2(t, x(t), u(t)) \equiv g^p((1-u)k, 1-v, h) - (n+\delta_2)h(t).$

The additional state variable x_3 is defined by

$$\frac{dx_3(t)}{dt} = f_3(t, x(t), u(t)) \equiv e^{-(\rho-n)t}U(c(t)), \ x_3(0) = 0.$$

Letting $\varepsilon = (t_0, t_1, x(t_0), x(t_1))$ denote the vector consisting of the initial time, terminal time, the initial state and the terminal state. The end conditions for the problem are given by

$$\phi_2(\varepsilon) = t_0 - 0 = 0,$$
$$\phi_3(\varepsilon) = t_1 - T = 0,$$
$$\phi_4(\varepsilon) = x_1(t_0) - k_0 = 0,$$
$$\phi_5(\varepsilon) = x_2(t_0) - h_0 = 0,$$
$$\phi_6(\varepsilon) = x_3(t_0) - 0 = 0.$$

The set of ε satisfying the end conditions $\phi_j(\varepsilon) = 0, j = 2, 3 \cdots 6$ is denoted by S. The performance index to be minimized is

$$I(x_0, u) = \phi_1(\varepsilon) = -x_3(t_1).$$

Let $F(t, x) = \{f(t, x, u) : u \in \Omega\}$ for each $(t, x) \in R^4$. $F(t, x)$ is the image of the control set Ω under the function $f(t, x, \cdot)$. If Ω is compact, $F(t, x)$ is compact since f is continuous. Here we need the additional assumption to guarantee the existence of optimal solution:

Assumption 3: Suppose that f is continuous; moreover, there exist positive constants c_1, c_2 such that

$$|f(t, x, u)| \le c_1(1 + |x| + |u|),$$
$$|f(t, x', u) - f(t, x, u)| \le c_2|x' - x|(1 + |u|),$$
$$\text{for all } t \in R, \ x, x' \in R^3, \text{ and } u \in \Omega.$$

Fleming and Rishel(1975) have proved the following theorem:

The Existence Theorem: Suppose that the assumption 3 holds; suppose moreover that

(a) F is not empty;

(b) Ω is compact;

(c) S is compact and ϕ is continuous on S;

(d) F(t,x) is convex for each $(t,x) \in R^4$.

Then there exists (x_0^*, u^*) minimizing $I(x_0, u)$ on F.

It is clear that F is not empty and Ω is compact. In addition, S is compact and ϕ is continuous. The functions $f_i(t,x,u), i = 1,2,3$ are concave with respect to the control variables so that their hypograph is convex since Ω is convex. This result is well-known. See, for instance, Mangasarian(1969). $F(t,x)$ is convex. Thus, the conditions (a) through (d) hold. It is also obvious that the right hand side of (5') and (6') is bounded and has suitable bounds on their partial derivatives, so that the assumption 3 is satisfied. Therefore, there exists an optimal solution that maximizes the objective function $J(k_0, h_0, u, v, c, T)$ on F.

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